

KdV6: An Integrable System

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Abstract

K^2S^2T [5] recently derived a new 6^{th} -order wave equation $KdV6$: $(\partial_x^2 + 8u_x\partial_x + 4u_{xx})(u_t + u_{xxx} + 6u_x^2) = 0$, found a linear problem and an auto-Bäcklund transformation for it, and conjectured its integrability in the usual sense. We prove this conjecture by constructing an infinite commuting hierarchy KdV_n6 with a common infinite set of conserved densities. A general construction is presented applicable to any bi-Hamiltonian system (such as all standard Lax equations, continuous and discrete) providing a nonholonomic perturbation of it. This perturbation is conjectured to preserve integrability. That conjecture is verified in a few representative cases: the classical long-wave equations, the Toda lattice (both continuous and discrete), and the Euler top.

1 Introduction

The theory of differential and difference Lax equations has been well understood by the middle of the 1980s, and no surprises have disturbed the contented tranquility of the subject ever since. Until now.

Recently, the 5 authors of [5] subjected to the Painlevé analysis the 6^{th} -order nonlinear wave equation

$$u_{xxxxx} + au_x u_{xxxx} + bu_{xx} u_{xxx} + cu_x^2 u_{xx} + du_{tt} + eu_{xxt} + fu_x u_{xt} + gu_t u_{xx} = 0, \quad (1.1)$$

where a, b, c, d, e, f and g are arbitrary parameters, and they have found 4 cases that pass the Painlevé test. Three of these were previously known, but the 4^{th} one turned out to be new (eqn (5) in [5]):

$$(\partial_x^2 + 8u_x\partial_x + 4u_{xx})(u_t + u_{xxx} + 6u_x^2) = 0. \quad (1.2)$$

This equation, as it stands, doesn't belong to any recognizable theory. The K^2S^2T convert it, in the variables $v = u_x$, $w = u_t + u_{xxx} + 6u_x^2$, into (eqn (12) in [5]):

$$v_t + v_{xxx} + 12vv_x - w_x = 0, \quad w_{xxx} + 8vw_x + 4wv_x = 0. \quad (1.3)$$

If integrable, this is a truly remarkable system: since $\{w = 0\}$ leaves only the unperturbed KdV itself, and the constrain on w is *differential*, what we have here is a *nonholonomic* deformation of the KdV ($= KdV_2$) equation. The 5 authors of [5] found a linear problem and an auto-Bäcklund transformation for their equation, but reported that they were unable to find higher symmetries on available computers, and asked if higher conserved densities and a Hamiltonian formalism exist for their equation.

All of these queries are resolved affirmatively below.

We now proceed to the general construction of nonholonomic perturbations of bi-Hamiltonian systems. Rescaling v and t in the equation (1.3), we get:

$$u_t = 6uu_x + u_{xxx} - w_x, \quad w_{xxx} + 4uw_x + 2u_xw = 0. \quad (1.4)$$

This can be converted into

$$u_t = B^1\left(\frac{\delta H_{n+1}}{\delta u}\right) - B^1(w) = B^2\left(\frac{\delta H_n}{\delta u}\right) - B^1(w), \quad B^2(w) = 0, \quad (1.5)$$

where

$$B^1 = \partial = \partial_x, \quad B^2 = \partial^3 + 2(u\partial + \partial u) \quad (1.6)$$

are the two standard Hamiltonian operators of the KdV hierarchy, $n = 2$, and

$$H_1 = u, \quad H_2 = u^2/2, \quad H_3 = u^3/3 - u_x^2/2, \dots \quad (1.7)$$

are the conserved densities.

And *that's it*. The ansatz (1.5) provides a nonholonomic deformation of *any* bi-Hamiltonian system. The question, naturally, is whether this ansatz is reasonable or an absurd phantasy. My answer is two-fold: (A) It is reasonable; (B) It is difficult, if not impossible, to prove integrability in general. The arguments are as follows.

(A) *Each* system (1.5) has an infinite sequence of H_m 's as its conserved densities:

$$\begin{aligned} \frac{dH_m}{dt} &\sim \frac{\delta H_m}{\delta u} \frac{\partial u}{\partial t} = \frac{\delta H_m}{\delta u} [B^2\left(\frac{\delta H_n}{\delta u}\right) - B^1(w)] \sim B^1\left(\frac{\delta H_m}{\delta u}\right)w = \\ &= B^2\left(\frac{\delta H_{m-1}}{\delta u}\right)w \sim -\frac{\delta H_{m-1}}{\delta u} B^2(w) = 0, \end{aligned} \quad (1.8)$$

where, as usual, $a \sim b$ means: $(a - b) \in \text{Im } \partial$ (a “trivial Lagrangian”).

(B) The above calculation is about the only one that can be reliably performed in the $\{u; w\}$ -picture, because the constraint $B^2(w) = 0$ is *nonholonomic*. Thus, if we proceed to develop the variational calculus in the $\{u; w\}$ -variables, we would be blocked, because the calculus works *only* when the factor $\Omega^1/\partial(\Omega^1)$ is a *free* module, where Ω^1 is the module of differential 1-forms (see [8]). Thus, the question of integrability: whether the flows (1.5) still commute between themselves, can not be answered in general with the modern tools. It *can* be answered for the KdV case (and I believe for all the standard differential Lax equations) through a subterfuge. To get a hint on how to proceed, we start in the next Section with the classical long-wave system

$$u_t = h_x + uu_x, \quad h_t = (uh)_x.$$

Section 3 is devoted to the KdV_n hierarchy (1.5,6) itself. Section 4 treats the Toda lattice and its continuous limit. The last section considers the classical Euler top.

Remark 1.9. The term *nonholonomic* is of a recent vintage, and seems to have been invented by Hertz, see [3].

2 The Classical Long-Wave Equations

The classical long wave system is bi-Hamiltonian (in fact, 3-Hamiltonian) [11], [6], [8]:

$$\begin{pmatrix} u \\ h \end{pmatrix}_t = \partial \begin{pmatrix} h + u^2/2 \\ uh \end{pmatrix} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \delta/\delta u \\ \delta/\delta h \end{pmatrix} \begin{pmatrix} u^2h + h^2 \\ 2 \end{pmatrix} = \quad (2.1a)$$

$$= \begin{pmatrix} 2\partial & \partial u \\ u\partial & h\partial + \partial h \end{pmatrix} \begin{pmatrix} \delta/\delta u \\ \delta/\delta h \end{pmatrix} \begin{pmatrix} uh \\ 2 \end{pmatrix}. \quad (2.1b)$$

Thus, its perturbation (1.5) is:

$$u_t = (h + u^2/2)_x - w_{2,x}, \quad h_t = (uh)_x - w_{1,x}, \quad (2.2a)$$

$$(2w_1 + uw_2)_x = 0, \quad uw_{1,x} + (h\partial + \partial h)(w_2) = 0. \quad (2.2b)$$

The first of the constraints in (2.2b) is resolvable, but the second one is not, and we seem to be stuck. The help comes from the missing from (2.1) (gravity) parameter g [1], rescaled away for mathematical simplicity (which was immaterial in the holonomic framework, but is fatal in the nonholonomic case):

$$\begin{pmatrix} u \\ h \end{pmatrix}_t = \partial \begin{pmatrix} gh + u^2/2 \\ uh \end{pmatrix} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \partial/\partial u \\ \partial/\partial h \end{pmatrix} \begin{pmatrix} u^2h + gh^2 \\ 2 \end{pmatrix} = \quad (2.3a)$$

$$= \begin{pmatrix} 2g\partial & \partial u \\ u\partial & h\partial + \partial h \end{pmatrix} \begin{pmatrix} \delta/\delta u \\ \delta/\delta h \end{pmatrix} \begin{pmatrix} uh \\ 2 \end{pmatrix}, \quad (2.3b)$$

so that now

$$u_t = (gh + u^2/2 - w_2)_x, \quad h_t = (uh - w_1)_x, \quad (2.4a)$$

$$(2gw_1 + uw_2)_x = 0, \quad uw_{1,x} + (h\partial + \partial h)(w_2) = 0. \quad (2.4b)$$

The constraint (2.4b) is *resolvable* as a regular series in g . This can be seen as follows. The first eqn in (2.4b) yields: $2gw_1 + uw_2 = \text{function of } t \text{ and } \epsilon \text{ only}$, and we rescale that function into 1:

$$2gw_1 + uw_2 = 1 \Rightarrow \quad (2.5)$$

$$w_2 = (1 - 2gw_1)/u \Rightarrow \quad (2.6)$$

$$w_{1,x} = -\frac{1}{u}(h\partial + \partial h)\frac{1}{u}(1 - 2gw_1) = \left(-\frac{h}{u^2}\right)_x + 2g\left(\frac{h}{u^2}\partial + \partial\frac{h}{u^2}\right)(w_1). \quad (2.7)$$

Set now

$$w_1 = \sum_{k=0}^{\infty} g^k z_k. \quad (2.8a)$$

Then

$$z_0 = -h/u^2, \quad z_{k+1} = 2\left(\frac{h}{u^2}\partial + \partial\frac{h}{u^2}\right)(z_k) \Rightarrow \quad (2.8b)$$

$$w_1 = -\sum_{k=0}^{\infty} \binom{2k+1}{k} g^k (h/u^2)^{k+1} = \frac{1}{2g} [1 - (1 - 4g\frac{h}{u^2})^{-1/2}] \Rightarrow \quad (2.9a)$$

$$w_2 = \frac{1}{u} (1 - 4g\frac{h}{u^2})^{-1/2} \Rightarrow \quad (2.9b)$$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \delta/\delta u \\ \delta/\delta h \end{pmatrix}(G), \quad G = \frac{u - \sqrt{u^2 - 4gh}}{2g}. \quad (2.10)$$

But G commutes with all the H_n 's, because

$$G_{hh}/G_{uu} = 2g/h = H_{n,hh}/H_{n,uu}, \quad \forall n \in \mathbb{Z}_{\geq 2}. \quad (2.11)$$

Thus, all the flows (1.5) commute also.

The workable approach to our general problem hence is this:

(\hat{A}) Rescale the variables u in (1.5) in such a way that the nonholonomic constrain $B^2(w) = 0$ becomes resolvable, hopefully in the form

$$w = \delta G / \delta u. \quad (2.11)$$

(\hat{B}) If then

$$\{G, H_n\} \sim 0, \quad \forall n, \quad (2.12)$$

then all the flows (1.5) commute between themselves. (See the end of Section 4 for more on this.)

Let's see now how this approach works for the KdV_n6 case.

Remark 2.13. Since the long-wave system is *three-Hamiltonian*, the nonholonomic construction applies not only to the pair (B^1, B^2) , but also to the pair (B^2, B^3) of the corresponding Hamiltonian structures. It's not clear how these two different perturbations are related.

Remark 2.14. N -component systems of hydrodynamical type (= 0-dispersion) are trivial for $N < 3$, but their nonholonomic perturbations are no longer so.

3 KdV_n6

We rescale ∂_t and ∂_x by ϵ . The KdV_n6 (1.4-7) becomes (1.5), now with

$$B^1 = \partial, \quad B^2 = \epsilon^2 \partial^3 + 2(u\partial + \partial u), \quad (3.1)$$

$$B^2(w) = \epsilon^2 w_{xx} + 2(u\partial + \partial u)(w) = 0. \quad (3.2)$$

To solve (3.2), set

$$w = \sum_{k=0}^{\infty} \epsilon^k w_k. \quad (3.3)$$

We get:

$$w_0 = u^{-1/2}, \quad w_1 = \dots, \quad (3.3)$$

and in fact

$$w = \frac{\delta G}{\delta u}, \quad G = \sum_{s=0}^{\infty} u^{1/2-s} \binom{1/2}{s} p_s, \quad \{G, H_n\} \sim 0, \quad \forall n, \quad (3.4)$$

where p_s are certain differential polynomials from the differential algebra $\mathbb{Q}[u^{(1)}, u^{(2)}, \dots][\epsilon]$. The proof is, unfortunately, rather long, and I omit it ([10]). I believe that similar rescaling works for the general differential Lax (= Gel'fand-Dickey) hierarchy, with the Lax operator

$$\mathcal{L} = u + \sum_{i=1}^N u_i (\epsilon \partial)^i, \quad u_N = 1, \quad u_{N-1} = 0, \quad N \in \mathbb{Z}_{\geq 3}, \quad (3.5)$$

but I haven't proved it. (The method was explained, for the case of the Burgers hierarchy, in my talk at the AMS meeting at Williams College in the fall of 2001.)

Remark 3.6. The situation becomes much more complicated when one passes to the *modified* Lax equations. For the KdV6 case, with the Miura map $u = \epsilon v_x - v^2$, one gets:

$$v_t = \epsilon^2 v_{xxx} - 6v^2 v_x + p, \quad (3.7a)$$

$$(\epsilon \partial - 2v)(p) = \partial(w), \quad (\epsilon \partial - 2v)\partial(\epsilon \partial + 2v)(w) = 0, \quad (3.7b)$$

so that one has a *pair* of nonholonomic constraints attached to *one* scalar field v .

4 The Toda Lattice

The Toda lattice is a classical mechanical system with the Hamiltonian

$$H = \sum_n \left(\frac{p_n^2}{2} + e^{q_{n+1} - q_n} \right). \quad (4.1)$$

In the variables

$$a_n = p_n, \quad b_n = \exp(q_{n+1} - q_n), \quad (4.2)$$

the motion equations become:

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix}_t &= \\ \begin{pmatrix} (1 - \Delta^{-1})(b) \\ b(\Delta - 1)(a) \end{pmatrix} &= \begin{pmatrix} 0 & (1 - \Delta^{-1})b \\ b(\Delta - 1) & 0 \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & (1 - \Delta^{-1})b \\ b(\Delta - 1) & 0 \end{pmatrix} \begin{pmatrix} \delta/\delta a \\ \delta/\delta b \end{pmatrix} \left(\frac{a^2}{2} + b \right) = \\ &= \begin{pmatrix} b \Delta - \Delta^{-1} b & a(1 - \Delta^{-1})b \\ b(\Delta - 1)a & b(\Delta - \Delta^{-1})b \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b \Delta - \Delta^{-1} b & a(1 - \Delta^{-1})b \\ b(\Delta - 1)a & b(\Delta - \Delta^{-1})b \end{pmatrix} \begin{pmatrix} \delta/\delta a \\ \delta/\delta b \end{pmatrix} (a), \end{aligned}$$

$$(4.3)$$

where Δ is the shift operator: $(\Delta f)(n) = f(n+1)$, all the equalities are understood as between functions of $n \in \mathbb{Z}$ (or $\mathbb{Z}/N\mathbb{Z}$), and (4.3) shows the first two (out of three) Hamiltonian structures, in the $\{a; b\}$ -variables, of the Toda lattice (see [7]).

The nonholonomic deformation ansatz (1.5) produces:

$$a_t = (1 - \Delta^{-1})(b - bw_2), \quad b_t = b(\Delta - 1)(a - w_1), \quad (4.4)$$

$$(b \Delta - \Delta^{-1} b)(w_1) + a(1 - \Delta^{-1})(bw_2) = 0, \quad (4.5a)$$

$$b(\Delta - 1)(aw_1 + (1 + \Delta^{-1})(bw_2)) = 0. \quad (4.5b)$$

The nonholonomic constrain (4.5), as it stands, is unresolvable. To proceed, we first rescale b into ϵb and then look for solutions regular in ϵ . We get:

$$(b \Delta - \Delta^{-1} b)(w_1) + a(1 - \Delta^{-1})(w) = 0, \quad (4.6a)$$

$$b(\Delta - 1)(aw_1 + \epsilon(1 + \Delta^{-1})(w)) = 0, \quad w := bw_2. \quad (4.6b)$$

(4.6b) implies that $aw_1 + \epsilon(1 + \Delta^{-1})(w) = \text{function of } t \text{ and } \epsilon \text{ only}$, and we rescale it into 1:

$$aw_1 + \epsilon(1 + \Delta^{-1})(w) = 1 \Rightarrow \quad (4.7)$$

$$\begin{aligned} w_1 &= \frac{1}{a}[1 - \epsilon(1 + \Delta^{-1})(w)] \Rightarrow \\ &-(1 - \Delta^{-1})(w) = \frac{1}{a}(b \Delta - \Delta^{-1} b) \frac{1}{a}[1 - \epsilon(1 + \Delta^{-1})(w)] = \\ &= \left(\frac{b}{aa^{(1)}} \Delta - \Delta^{-1} \frac{b}{aa^{(1)}} \right) [1 - \epsilon(1 + \Delta^{-1})(w)] = (1 - \Delta^{-1})(c) + \\ &-\epsilon(c \Delta - \Delta^{-1})(1 + \Delta^{-1})(w), \quad c := b/aa^{(1)}, \end{aligned} \quad (4.8)$$

where

$$q^{(s)} := \Delta^s(q), \quad s \in \mathbb{Z}. \quad (4.9)$$

Setting

$$w = - \sum_{k=0}^{\infty} z_k \epsilon^k, \quad (4.10)$$

we find:

$$(1 - \Delta^{-1})(z_{k+1}) = (c \Delta - \Delta^{-1} c)(1 + \Delta^{-1})(z_k), \quad z_0 = c, \quad k \in \mathbb{Z}_{\geq 0}. \quad (4.11)$$

The latter equation, being nonlocal, looks rather impenetrable; it's not even clear if it's solvable. So let's pass to the continuous limit to see what the situation is in simpler

circumstances. The previous formulae become:

$$\begin{pmatrix} a \\ b \end{pmatrix}_t = \begin{pmatrix} b_x \\ ba_x \end{pmatrix} = \begin{pmatrix} 0 & \partial b \\ b\partial & 0 \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & \partial b \\ b\partial & 0 \end{pmatrix} \begin{pmatrix} \delta/\delta u \\ \delta/\delta b \end{pmatrix} \left(\frac{a^2}{2} + b \right) = \quad (4.12)$$

$$= \begin{pmatrix} b\partial + \partial b & a\partial b \\ b\partial a & 2b\partial b \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b\partial + \partial b & a\partial b \\ b\partial a & 2b\partial b \end{pmatrix} \begin{pmatrix} \delta/\delta a \\ \delta/\delta b \end{pmatrix} (a),$$

$$a_t = (\epsilon b(1 - w_2))_x, \quad b_t = b(a - w_1)_x, \quad (4.13)$$

$$(b\partial + \partial b)(w_1) + aw_x = 0, \quad w := bw_2, \quad (4.14a)$$

$$b\partial(aw_1 + 2\epsilon w) = 0. \quad (4.14b)$$

(4.14b) resolves into

$$aw_1 + 2\epsilon w = 1 \Rightarrow \quad (4.15)$$

$$w_1 = (1 - 2\epsilon w)/a \Rightarrow$$

$$\begin{aligned} -w_x &= \frac{1}{a}(b\partial + \partial b)\frac{1}{a}(1 - 2\epsilon w) = \left(\frac{b}{a^2}\partial + \partial\frac{b}{a^2}\right)(1 - 2\epsilon w) = \\ &= c_x - 2\epsilon(c\partial + \partial c)(w), \quad c := b/a^2. \end{aligned} \quad (4.16)$$

Setting $w = -\sum_{k=0}^{\infty} z_k \epsilon^k$ again (4.10), we find:

$$z_{k+1,x} = 2\epsilon(c\partial + \partial c)(z_k), \quad z_0 = c \Rightarrow \quad (4.17)$$

$$z_k = \binom{2k+1}{k} c^{k+1}, \quad (4.18)$$

and the calculation identical to that of §2 shows that (now with $\epsilon = 1$)

$$w_1 = \eta^{-1/2}, \quad \eta := a^2 - 4b, \quad (4.19a)$$

$$w_2 = \frac{1 - a\eta^{-1/2}}{2b}. \quad (4.19b)$$

Since

$$\frac{\partial w_1}{\partial b} = \frac{\partial w_2}{\partial a} = 2\eta^{-3/2}, \quad (4.20)$$

there exists a Hamiltonian $G = G(a, b)$ such that

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \delta/\delta a \\ \delta/\delta b \end{pmatrix} (G), \quad (4.21)$$

and this G commutes with all the conserved densities H_m 's of the continuous Toda flow (4.12), because

$$G_{aa}/(b\partial_b)^2(G) = \frac{1}{b} = H_{m,aa}/(b\partial_b)^2(H_m), \quad \forall m. \quad (4.22)$$

Thus, the continuous limit picture is manageable. Back to the discrete play, the equation (4.11.) There exist no general methods to handle nonlocal recurrences such as (4.11)

save for the method of bi-Hamiltonian systems (see [9].) So, we first move (4.11) into a skewsymmetric form, by applying from the left the operator $(1 + \Delta)$, resulting in:

$$(\Delta - \Delta^{-1})(z_{k+1}) = (1 + \Delta)(c \Delta - \Delta^{-1} c)(1 + \Delta^{-1})(z_k), \quad z_0 = c. \quad (4.23)$$

Unfortunately, the form (4.23), *as it stands*, is *not* of the bi-Hamiltonian character, because, e.g.,

$$z_1 = c^{(1)}c + cc + cc^{(1)} \notin \text{Im}(\delta), \quad (4.24)$$

i.e., z_1 is not $\delta H / \delta c$ for any H . This however, shouldn't be the end of the story, and it isn't. The help comes from the observation that

$$z_1/c = \frac{\delta}{\delta c}(c^2 + c^{(-1)}c + cc^{(1)})/2, \quad (4.25a)$$

$$D(z_2/c) = [D(z_2/c)]^\dagger, \quad (4.25b)$$

so that $z_2/c \in \text{Im}(\delta)$; here $D(\cdot)$ is the Fréchet derivative of (\cdot) . This strongly suggests that we set

$$z_k = c\rho_k, \quad k \in \mathbb{Z}_{\geq 0}, \quad (4.26)$$

multiply eqn (4.23) from the left by c , and rewrite (4.23) as

$$[c(\Delta - \Delta^{-1})c](\rho_{k+1}) = [c(1 + \Delta)(c \Delta - \Delta^{-1} c)(1 + \Delta^{-1})c](\rho_k), \quad \rho_0 = 1. \quad (4.27)$$

Miraculously, and for no discernible reason:

(a) The matrix (in fact, scalar)

$$b^2 = c(1 + \Delta)(c \Delta - \Delta^{-1} c)(1 + \Delta^{-1})c \quad (4.28)$$

is Hamiltonian;

(b) The pair of Hamiltonian matrices, b^2 (4.28) and

$$b^1 = c(\Delta - \Delta^{-1})c \quad (4.29)$$

form a Hamiltonian pair. The bi-Hamiltonian theory then guarantees the existence of a sequence of Hamiltonians $\{h_m\}$ such that

$$\rho_m = \delta h_m / \delta c, \quad m \in \mathbb{Z}_{\geq 0}. \quad (4.30)$$

Thus, our constrain (4.8) has been resolved:

$$w = bw_2 = - \sum_{k=0}^{\infty} \epsilon^k c(\delta h_k / \delta c), \quad c = b/aa^{(1)}. \quad (4.31)$$

Setting

$$h = \sum_{k=0}^{\infty} \epsilon^k h_k, \quad (4.32)$$

and noticing that

$$\frac{\delta}{\delta a} = -\frac{1}{a}(1 + \Delta^{-1})c \frac{\delta}{\delta c}, \quad \frac{\delta}{\delta b} = \frac{1}{aa^{(1)}} \frac{\delta}{\delta c}, \quad (4.33)$$

we see that

$$-w_2 = \frac{1}{b} \sum_{k=0}^{\infty} c \epsilon^k \frac{\delta h_k}{\delta c} = \frac{1}{a} \frac{1}{a^{(1)}} \frac{\delta h}{\delta c} = \frac{\delta h}{\delta b}, \quad (4.34a)$$

$$w_1 = \frac{1}{a} - \epsilon \frac{1}{a} (1 + \Delta^{-1})(-c \frac{\delta h}{\delta c}) = \frac{1}{a} - \frac{\delta h}{\delta a} = \frac{\delta}{\delta a} (\log a - h). \quad (4.34b)$$

Thus,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \delta/\delta a \\ \delta/\delta b \end{pmatrix} (\log a - h), \quad (4.35)$$

and the last step now is to show that $\log a - h$ commutes with all the conserved densities H_m 's of the full Toda lattice (4.3). But this is true *in general*:

(\hat{C}) Suppose that the constrain $B^2(w) = 0$ is (1.5) has been resolved as

$$w = \frac{\delta G}{\delta u}, \quad \text{some } G. \quad (4.36)$$

Then

$$\begin{aligned} \{G, H_m\} &\sim \frac{\delta H_m}{\delta u^t} B^1 \left(\frac{\delta G}{\delta u} \right) \sim - \left[B^1 \left(\frac{\delta H_m}{\delta u} \right) \right]^t \frac{\delta G}{\delta u} = \\ &= - \left[B^2 \left(\frac{\delta H_{m-1}}{\delta u} \right) \right]^t \frac{\delta G}{\delta u} \sim \frac{\delta H_{m-1}}{\delta u^t} B^2 \left(\frac{\delta G}{\delta u} \right) = \frac{\delta H_{m-1}}{\delta u^t} B^2(w) = 0. \end{aligned} \quad (4.37)$$

Thus, our prescription for analyzing the nonholonomic deformation, stated at the end of Section 2, works perfectly provided the nonholonomic perturbation w is a variational derivative (4.36).

Remark 4.38. With hindsight, one readily sees that the bi-Hamiltonian pair b^1 (4.29) and b^2 (4.28) describes the classical Volterra lattice

$$\dot{c} = c(c^{(1)} - c^{(-1)}), \quad (4.39)$$

see [7].

Remark 4.40. The nonholonomic perturbation (1.5) of the Volterra lattice (4.39),

$$\dot{c} = c(\Delta - \Delta^{-1})c(1 - w), \quad (c \Delta - \Delta^{-1}c)(1 + \Delta^{-1})(w) = 0, \quad (4.41)$$

is not simplified by the rescaling $c \mapsto \epsilon c$, but it does so upon the rescaling $c \mapsto 1 + \epsilon c$ around the *stationary* solution $\{c = 1\}$ of the Volterra lattice.

5 The Euler Top

The constrain $B^2(w) = 0$ is, in general, nonholonomic only for systems which are either differential or difference on \mathbb{Z} , i.e., for dimensions 1 and “1/2”. In 0 dimensions, i.e., in Classical Mechanics with a *finite* number of degrees of freedom, the constrain $B^2(w) = 0$ becomes *holonomic*. Thus, e.g., there would be no problem in resolving that constrain for the *periodic* Toda lattice on $\mathbb{Z}/N\mathbb{Z}$.

Let’s see how this works in practice, for the simplest possible case, the $\mathfrak{so}(3)$ Euler top:

$$\dot{x}_1 = \alpha_1 x_2 x_3, \quad \dot{x}_2 = \alpha_2 x_3 x_1, \quad \dot{x}_3 = \alpha_3 x_1 x_2, \quad (5.1)$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. I follow the notation and results of a short and concise presentation in [12]. Let’s agree that (ijk) stands for an even permutation of (123) . Equations (5.1) can be rewritten as

$$\dot{x}_i = \alpha_i x_j x_k, \quad (5.2)$$

with $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ being an arbitrary but fixed vector of parameters. If $\mathbf{c} = (c_1, c_2, c_3)$ is another vector in \mathbb{R}^3 , then

$$H_c = \frac{1}{2} \sum_{i=1}^3 c_i x_i^2 \quad (5.3)$$

is an integral of (5.1) iff

$$(\alpha, \mathbf{c}) = 0, \quad (5.4)$$

so that the space of integrals is two-dimensional.

Now, for any vector $\gamma \in \mathbb{R}^3$, consider the Poisson brackets

$$\{x_i, x_j\}^{(\gamma)} = \gamma_k x_k. \quad (5.5)$$

Then

$$\{x_i, H_c\}^{(\gamma)} = c_j x_j \gamma_k x_k - c_k x_k \gamma_j x_j = (c_j \gamma_k - c_k \gamma_j) x_j x_k, \quad (5.6)$$

so that the conditions

$$\alpha = \mathbf{c} \times \gamma \Rightarrow (\alpha, \mathbf{c}) = 0, \quad (5.7)$$

guarantee that the motion equations (5.1) are Hamiltonian in the Hamiltonian structure (5.5) with the Hamiltonian (5.3). So, let’s choose vectors β and γ such that $\{\beta, \gamma, \alpha/|\alpha|\}$ form a right orthonormal basis. Set B^1 and B^2 corresponding to β and γ , respectively:

$$-B^1 = \begin{pmatrix} 0 & -\beta_3 x_3 & -\beta_2 x_2 \\ \beta_3 x_3 & 0 & \beta_1 x_1 \\ \beta_2 x_2 & -\beta_1 x_1 & 0 \end{pmatrix}, \quad -B^2 = \begin{pmatrix} 0 & -\gamma_3 x_3 & -\gamma_2 x_2 \\ \gamma_3 x_3 & 0 & \gamma_1 x_1 \\ \gamma_2 x_2 & -\gamma_1 x_1 & 0 \end{pmatrix}. \quad (5.8)$$

The constrain $B^2(w) = 0$ becomes:

$$\mathbf{w} \times \mathbb{X}^{(\gamma)} = 0, \quad \mathbb{X}^{(\gamma)} := (\gamma_1 x_1, \gamma_2 x_3, \gamma_3 x_3), \quad (5.9)$$

so that

$$\mathbf{w} = \text{const} \mathbb{X}^{(\gamma)}, \quad \text{const} = \text{const}(t), \quad (5.10)$$

and the perturbed motion equations (1.5) become:

$$\dot{x}_i = \alpha_i x_j x_k - \text{const}(\mathbb{X}^{(\beta)} \times \mathbb{X}^{(\gamma)})_i. \quad (5.11)$$

But

$$(\mathbb{X}^{(\beta)} \times \mathbb{X}^{(\gamma)})_i = (\boldsymbol{\beta} \times \boldsymbol{\gamma})_i x_j x_k = \frac{1}{|\boldsymbol{\alpha}|} \alpha_i x_j x_k \quad (5.12)$$

Thus, finally, the perturbed top equations are

$$\dot{x}_i = \text{const}' \alpha_i x_j x_k, \quad (5.13)$$

so that the overall effect of the perturbation amounts to the time rescaling of the original top. This is reminiscent of the general Chaplygin theorem identifying some special nonholonomic systems with the time-rescaled Hamiltonian ones (see [4,3].)

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